

3.3 | 연습문제 정답

01. 2

02. $\frac{2}{\pi}$

03. $y' = \sin^{-1}x$

04. $y' = \tan^{-1}x$

05. $y' = \frac{\sec^2\left(\frac{x}{2}\right)}{\sqrt{2}\left(2 + \tan^2\left(\frac{x}{2}\right)\right)}$

06. $y' = \frac{2}{1-x^2} + \frac{2x \sin^{-1}x}{(1-x^2)\sqrt{1-x^2}} - \frac{x}{1-x^2}$

07. $y' = -\frac{x}{|x|\sqrt{1-x^2}}$

08. $y' = \frac{1}{2}$

09. $y' = 2x \tanh^{-1}(-x+6) + \frac{x^2}{x^2-12x+35}$

10. $y' = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x^2+2x}}$

11. (1) $x \neq 0$ 인 모든 실수에 대하여 미분가능하다.
 (2) $x = 0$ 에서 미분불가능이다.

12. 수학적 귀납법을 이용하여 증명하면

(1) $n = 0$ 일 때

$$y' = \frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2} \text{ 이고}$$

$$\frac{1}{1+x^2} = \frac{1}{1+\tan^2 y} = \frac{1}{\sec^2 y} = \cos^2 y = \sin\left(y + \frac{\pi}{2}\right) \cos y \text{ 이므로}$$

$$y' = \sin\left(y + \frac{\pi}{2}\right) \cos y \text{ 이 성립한다.}$$

(2) $n = k-1$ 일 때 $y^{(k)} = (k-1)! \sin\left(ky + \frac{\pi}{2}k\right) \cos^k y$ 가 성립한다고 가정하자.

(3) $n = k$ 이면

$$\begin{aligned} y^{(k+1)} &= \frac{dy^{(k)}}{dx} = \frac{d}{dx} (k-1)! \left[\sin\left(ky + \frac{\pi}{2}k\right) \cos^k y \right] \\ &= (k-1)! \left[\cos\left(ky + \frac{\pi}{2}k\right) (k) (y') \cos^k y + \sin\left(ky + \frac{\pi}{2}k\right) (k) (\cos^{k-1} y) (-\sin y) (y') \right] \\ &= k! \sin\left((k+1)y + (k+1)\frac{\pi}{2}\right) \cos^{k+1} y \end{aligned}$$

13. $\frac{d}{dx} \left\{ \frac{1}{1 + [\sin^{-1} f(x)]^2} \right\} = \frac{-2[\sin^{-1} f(x)] \cdot f'(x)}{(1 + [\sin^{-1} f(x)]^2)^{5/2}}$

14. $(\tan^{-1}(f(2x)))' = \frac{2f'(2x)}{1 + (f(2x))^2}$

15. $y' = \cos(n \sin^{-1} x) \left(\frac{n}{\sqrt{1-x^2}} \right)$
 $\Rightarrow y'' = -\sin(n \sin^{-1} x) (n^2) \left(\frac{1}{\sqrt{1-x^2}} \right)^2$

$$+\cos(n\sin^{-1}x)(n)\left(-\frac{1}{2}\right)(1-x^2)^{-\frac{3}{2}}(-2x)$$

$$(1) (1-x^2)y'' = -\sin(n\sin^{-1}x)(n^2) + \cos(n\sin^{-1}x)(n)(1-x^2)^{-\frac{1}{2}}(x)$$

$$(2) -xy' = \cos(n\sin^{-1}x)(n)(1-x^2)^{-\frac{1}{2}}(-x)$$

$$(3) n^2y = n^2\sin(n\sin^{-1}x)$$

$$\therefore (1), (2), (3)\text{에 의하여 } (1-x^2)y'' - xy' + n^2y = 0$$

16. (1) $2y = 2e^{-x}\cos x$

$$(2) y' = -e^{-x}\cos x + e^{-x}(-\sin x) \Rightarrow 2y' = -e^{-x}(2\cos x + 2\sin x)$$

$$(3) y'' = e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x) = 2e^{-x}\sin x$$

$$\therefore (1), (2), (3)\text{에 의하여 } y'' + 2y' + 2y = 0$$

17. (a) $a^2x = t \Rightarrow a^2 \frac{dx}{dt} = 1, y' = \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = a^2 \frac{dy}{dt},$

$$y'' = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{a^2 \frac{d^2y}{dt^2}}{\frac{1}{a^2}}$$

$$\therefore xy'' + by' - a^2y = 0 \Leftrightarrow \left(\frac{t}{a^2} \right) \left(\frac{a^2 \frac{d^2y}{dt^2}}{\frac{1}{a^2}} \right) + ba^2 \frac{dy}{dt} - a^2y = 0$$

$$\Leftrightarrow a^2t \frac{d^2y}{dt^2} + ba^2 \frac{dy}{dt} - a^2y = 0$$

$$\Leftrightarrow t \frac{d^2y}{dt^2} + b \frac{dy}{dt} - y = 0$$

$$(b) t = -\frac{u^2}{4} \Rightarrow \frac{du}{dt} = -\frac{2}{u}, b = n+1 \Rightarrow \frac{dy}{du} = \frac{dy}{dt} \frac{dt}{du} = -\frac{u}{2} \frac{dy}{dt}$$

$$\frac{d^2 y}{du^2} = \frac{d\left(\frac{dy}{du}\right)}{\frac{du}{dt}} = \frac{\frac{d^2 y}{dt^2} \left(-\frac{u}{2}\right) + \frac{dy}{dt} \left(-\frac{1}{2}\right) \frac{du}{dt}}{-\frac{2}{u}} = \frac{u^2}{4} \frac{d^2 y}{dt^2} - \frac{1}{2} \frac{dy}{dt} = \frac{u^2}{4} \frac{d^2 y}{dt^2} + \frac{1}{u} \frac{dy}{du}$$

$$\therefore t \frac{d^2 y}{dt^2} + b \frac{dy}{dt} - y = 0$$

$$\Leftrightarrow \left(-\frac{u^2}{4}\right) \left(\frac{d^2 y}{du^2} - \frac{1}{u} \frac{dy}{du}\right) \left(\frac{4}{u^2}\right) + (n+1) \left(-\frac{2}{u} \frac{dy}{du}\right) - y = 0$$

$$\Leftrightarrow -\frac{d^2 y}{du^2} + \frac{1}{u} \frac{dy}{du} - \frac{2(n+1)}{u} \frac{dy}{du} - y = 0$$

$$\Leftrightarrow u \frac{d^2 y}{du^2} + (2n+1) \frac{dy}{du} + uy = 0$$

$$(c) \ y = \frac{z}{u^n} \Rightarrow z = yu^n, \quad \frac{dz}{du} = \frac{dy}{du} u^n + ynu^{n-1},$$

$$\frac{d^2 z}{du^2} = \frac{d^2 y}{du^2} u^n + 2 \frac{dy}{du} nu^{n-1} + n(n-1)yu^{n-2}$$

$$\therefore u \frac{d^2 y}{du^2} + (2n+1) \frac{dy}{du} + uy$$

$$= u \left(\frac{d^2 z}{du^2} - 2 \frac{dy}{du} nu^{n-1} - n(n-1)yu^{n-2} \right) \left(\frac{1}{u^n} \right) + (2n+1) \left(\frac{dz}{du} - ynu^{n-1} \right) \left(\frac{1}{u^n} \right) + u \frac{z}{u^n} = 0$$

$$\Leftrightarrow u \frac{d^2 z}{du^2} - 2nu^n \frac{dy}{du} - n(n-1)yu^{n-1} + (2n+1) \frac{dz}{du} - n(2n+1)yu^{n-1} + uz = 0$$

$$\Leftrightarrow u \frac{d^2 z}{du^2} - 2nu^n \left(\frac{dz}{du} - ynu^{n-1} \right) \left(\frac{1}{u^n} \right) + (2n+1) \frac{dz}{du} - 3n^2 yu^{n-1} + uz = 0$$

$$\Leftrightarrow u^2 \frac{d^2 z}{du^2} + u \frac{dz}{du} + (u^2 - n^2)z = 0$$